Above and below primitivity

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A structure on Ω is trivial if it is invariant under the whole symmetric group on Ω; it is non-trivial otherwise. Many permutation group properties are defined by requiring that there is no non-trivial structure of some type admitting the group. Several examples follow.

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Primitivity is probably the most important such property, and was introduced by Galois.

O'Nan–Scott; basic groups

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A permutation group G on Ω is basic if it is primitive and there is no non-trivial G-invariant Hamming graph on Ω. Now one half of the celebrated O'Nan–Scott Theorem describes the structure of non-basic primitive groups in terms of their socle (product of the minimal normal subgroups).

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The second half is on the following slide.

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The O'Nan–Scott Theorem is the basis of most recent results on primitive groups. In the first two cases, we have a geometric structure to study; in the third, we have to use the Classification of Finite Simple Groups, and detailed knowledge about these groups.

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It follows that a synchronizing permutation group is transitive (take a complete graph on an orbit); primitive (take an equivalence relation as graph); and basic (the Hamming graph $H(n, q)$ have clique number and chromatic number q).

Coherent configurations and association schemes

Coherent configurations were introduced by Donald Higman to study permutation groups (the history is really a bit more complicated). A coherent configuration on Ω is a partition of $\Omega \times \Omega$ such that the diagonal is a union of parts, the converse of a part is a part, and for any three parts A, B, C, if $(x, y) \in C$, the number of $z \in \Omega$ such that $(x, z) \in A$ and $(z, y) \in B$ depends only on A, B, C and not on x, y. The orbits on Ω^2 of a permutation group on Ω form a coherent configuration.

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A transitive permutation group G on Ω is AS-free if there is no non-trivial G-invariant association scheme on Ω. Thus, 2-homogeneous permutation groups are AS-free; but there are others.

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Diagonal groups pose a real puzzle. Let d be the number of simple factors in the socle. If $d = 2$, they preserve the conjugacy class scheme of the simple group; if $d = 3$, they preserve the Latin square graph of its Cayley table. Beyond this, despite a number of attempts, we just don't know!

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So this is also a strengthening of primitivity; and 2-transitive groups have the RCP, so it lies below 2-transitivity.
Surprisingly, this is also connected with transformation semigroups:

Theorem

G has the road closure property if and only if, for any map t on Ω with rank 2, the transformation monoid $\langle G, t \rangle \setminus G$ is idempotent-generated.

The current state of knowledge

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- For diagonal groups, there is an iterative construction: the group permuting the factors must be primitive, and it fails RCP if and only if the whole group does.
- All known almost simple primitive groups which fail RCP have two or three blocks of imprimitivity on edges, but we can't show this is necessarily so.

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- The join or supremum $\Pi_1 \vee \Pi_2$: form a graph in which two points are joined if they lie together in a part of Π_1 or a part of Π_2; the parts of the join are the connected components of this graph.

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Theorem

If a lattice L of partitions has the property that all pairs of partitions commute, then the lattice is modular.

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- In this correspondence, two partitions commute if and only if the corresponding subgroups commute (in the sense $HK = KH$).

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A permutation group has the PB property if it preserves a poset block structure.

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The second part is an extension of the classical Krasner–Kaloujnine Theorem. Rosemary will discuss this.

References

- J. Araújo, P. J. Cameron and B. Steinberg, Between primitive and 2-transitive: Synchronization and its friends, *Europ. Math. Soc. Surveys* **4** (2017), 101–184; doi: `10.4171/EMSS/4-2-1`.
- Marina Anagnostopoulou-Merkouri, R. A. Bailey and Peter J. Cameron, Permutation groups, partition lattices and block structures, *Forum of Mathematics Sigma* **13**:180 (2025), 32pp.; doi: `10.1017/fms.2025.10126`.

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- J. Araújo, P. J. Cameron and B. Steinberg, Between primitive and 2-transitive: Synchronization and its friends, *Europ. Math. Soc. Surveys* **4** (2017), 101–184; doi: 10.4171/EMSS/4-2-1.
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… for your attention.