Ramsey's theorem and topological dynamics

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In each case, Ramsey guarantees that, if the party is large enough, then we can find the set we are looking for.

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We are interested in classes of such structures (with the same named relations). We always assume that our classes are hereditary, that is, closed under taking induced substructures (formed by picking a subset and all instances of the relation within it). For example, if we are dealing with graphs, we take a set of vertices, and all the edges it contains.

Ramsey classes

Guided by Ramsey's theorem, we say that a class $\mathcal{C}$ of finite structures is a Ramsey class if, for any $A, B \in \mathcal{C}$, there is a $C \in \mathcal{C}$ such that, if the embeddings of A into C are coloured red and blue, then there is a copy of B in C all of whose embedded As have the same colour.

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In the party problem, A is a set of size 2, B a set of size 3, and we can take C to be a set of size 6.

Fraïssé classes

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For example, finite ordered sets and finite graphs form Fraïssé classes; their Fraïssé limits are respectively the rational numbers (as ordered set) and the Erdős–Rényi random graph.

Structure of Ramsey classes

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In Nešetřil's examples, the rigidity was enforced by having a total order as one of the relations.

A question

At the time, I constructed a completely different Fraïssé class of rigid structures, by superimposing a tournament (whose symmetry group has odd order) with a ternary relation derived from binary trees (whose group has 2-power order). There is no total order in sight; could such a class be a Ramsey class?

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This question was answered by a remarkable theorem proved by Kechris, Pestov and Todorčević, connecting Ramsey theory with topological dynamics.

Briefly, there is a natural topology on the symmetric group of countable degree (the topology of pointwise convergence); its closed subgroups are precisely the automorphism groups of relational structures, and so are themselves topological groups.

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The automorphism group of the Fraïssé limit of a nontrivial Ramsey class is extremely amenable.

This answers my question. For the set of all total orders on a countable set has a natural topology, and is compact; and the symmetric group acts continuously on it. So, if $\mathcal{C}$ is a nontrivial Ramsey class with Fraïssé limit M, then $\mathrm{Aut}(M)$ fixes a total order; its restriction to any finite subset gives a total order on that set fixed by its automorphisms, showing that these objects must be rigid.

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Further questions remain, but that's enough for now …