Synchronization

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Automata

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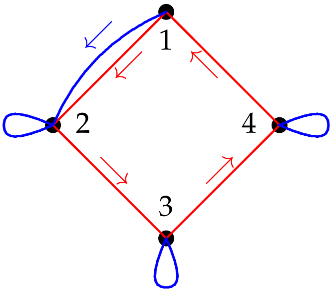
Research has been driven by the Černý conjecture, which states that if an n-state automaton is synchronizing, it has a reset word of length at most $(n-1)^2$.

An example



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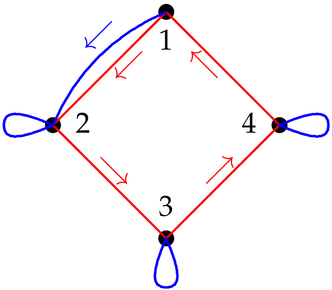

	B	R	R	R	B	R	R	R	B
1	2	3	4	1	2	3	4	1	2
2	2	3	4	1	2	3	4	1	2
3	3	4	1	2	2	3	4	1	2
4	4	1	2	3	3	4	1	2	2

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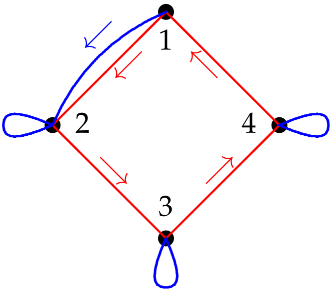

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So BRRRBRRRB is a reset word (and is in fact the shortest).

From automata to transformation monoids

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Since reading a sequence of symbols corresponds to composing these maps, the maps which can be produced by an automaton form a monoid (the identity transition corresponds to reading the empty word). In our example, the generators of the monoid are

$$\mathrm{R} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \end{pmatrix}.$$

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The preceding slide shows that the monoid associated with this automaton contains $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}$.

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> *Which permutation groups G on Ω have the property that, for any transition t which is not a permutation, the monoid $\langle G, t \rangle$ is synchronizing?*

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> *Which permutation groups G on Ω have the property that, for any transition t which is not a permutation, the monoid $\langle G, t \rangle$ is synchronizing?*

Since a permutation group of degree greater than 1 cannot be synchronizing as a monoid, we borrow the term "synchronizing" to describe permutation groups with this property.

From permutation groups to graphs

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- It is transitive (that is, has a single orbit). (If there is more than one orbit, take the complete graph on a single orbit.)
- It is primitive (that is, preserves no non-trivial equivalence relation). (If there is a non-trivial equivalence relation, take the union of complete graphs on the equivalence classes.)

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Other properties of permutation groups were investigated in my paper with João and Ben. One is worth discussing here.

Separation

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Thus “separating” implies “synchronizing”, but it is easier to check, since clique number is easier to compute than chromatic number. (Both problems are NP-hard, but parametrized complexity gives us ways of making sense of this statement: testing clique size k is in the complexity class $W[1]$, but no $W[1]$ algorithm for chromatic number is known.)

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I give an example on the next slide.

An example

Consider the symmetric group S_n acting on k-sets. Mohammed Aljohani, John Bamberg and I conjectured that, for $n \geq f(k)$, the only way that a G-invariant graph to be non-separating is, up to complementation, that two k-sets are joined if they meet in at most $t - 1$ points for some t; then a clique, coclique pair satisfying the equality must consist of the blocks of a Steiner system $S(t, k, n)$ for some n and an Erdős–Ko–Rado set (all the k-sets containing a fixed t-set).

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The group will fail to be synchronizing if and only if there is a partition of the k-sets into Steiner systems. We are still waiting for the design theorists to prove this!

Core automata and transducers

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If you want to know more, there are two sessions tomorrow: on semigroups and automata, and on Thompson groups.

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A folding of an automaton digraph is an equivalence relation on the set of states with the property that, if v_1 and w_1 are equivalent and there is an arc labelled a from v_1 to v_2, then there is a state w_2 and an arc labelled a from v_2 to w_2.

Collapsing the equivalence classes to single vertices, we get a smaller automaton digraph.

De Bruijn digraphs

The vertices of the de Bruijn digraph $G(k, n)$ are the words of length n over an alphabet of size k, with an arc from v to w if w is obtained from v by deleting the first letter and adding a letter at the end.

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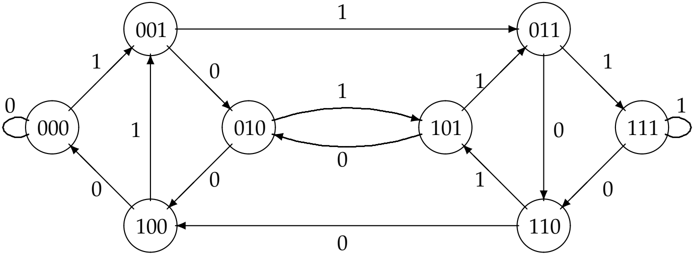

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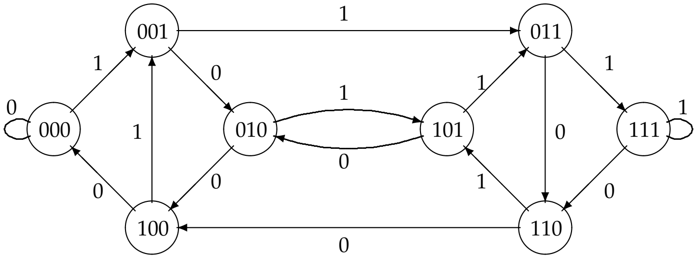

Theorem

An automaton is strongly synchronizing with word length n if and only if it is obtained by folding $G(k,n)$.

The rational group

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Note that the set of infinite strings over A has a natural metric, with respect to which it is homeomorphic to the Cantor set, and the rational group acts as homeomorphisms.

The Higman–Thompson groups

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It is approximately true that, if S is a Higman–Thompson group, then the outer automorphism group of S is the group of all elements of the GNS rational group generated by invertible core transducers T with the property that T and its inverse are both strongly synchronizing.

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