The countable homogeneous poset

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"On homogeneous graphs and posets"
by Jan Hubička and Jaroslav Nešetřil
Charles University, Prague, Czech Republic.

The random graph

Erdős–Rényi Theorem There is a unique countable graph R with the property that a random countable graph X (obtained by choosing edges independently with probability $1/2$) satisfies $\mathrm{Prob}(X \cong R) = 1$

The graph R is

- *universal*: every finite or countable graph is embeddable in R.

- *homogeneous*: any isomorphism between finite subgraphs of R extends to an automorphism of R.

These two properties characterise R.

Recognising R

R is the unique countable graph with the property that, for any finite graphs G and H with $G \subseteq H$, every embedding of G in R extends to an embedding of H.

It is enough to require this when $|H| = |G| + 1$: givn disjoint finite sets M_0, M_1 of vertices, there is a vertex x joined to all vertices of M_1 and none of M_0.

A similar characterisation holds for any countable homogeneous relational structure. Such a structure is determined by its class of finite substructures, this class having the amalgamation property (*Fraïssé's Theorem*).

Constructions of R

- Vertex set is a countable model of set theory; $x \sim y$ if $x \in y$ or $y \in x$.

- Vertex set is $\mathbb{N}$; $x \sim y$ if the xth binary digit of y is 1 (or *vice versa*).

- Vertex set is the set of primes congruent to 1 mod 4; $x \sim y$ if x is a quadratic residue mod y.

- Vertex set is $\mathbb{Z}$; $x \sim y$ if the $|x-y|$th term in a fixed universal binary sequence is 1.

Motivation

The Erdős–Rényi theorem is a non-constructive existence proof for R. This can be re-formulated in terms of Baire category instead of measure; in this form it applies to all countable homogeneous relational structures.

However, an explicit construction can give us more information.

For example, the fourth construction given earlier shows that R admits cyclic automorphisms (and, indeed, that the conjugacy classes of cyclic automorphisms of R are parametrised by the universal binary sequences).

Models of set theory

In showing that the first construction above gives R, we do not need all the axioms of ZFC: only the empty set, pairing, union, and foundation axioms.

Sketch proof: Let M_0 and M_1 be disjoint finite sets. Let $x = M_1 \cup \{y\}$, where y is chosen so that it is not in M_0 or in a member of a member of M_0. (This ensures that $z \notin x$ and $x \notin z$ for all $z \in M_0$.) Then x is joined to everything in M_0 and nothing in M_1.

In particular, the Axiom of Infinity is not used. Now there is a simple model of *hereditarily finite set theory*, satisfying the negation of the axiom of infinity: the ground set is $\mathbb{N}$, and $x \in y$ if the xth binary digit of y is 1. Thus the second construction is a special case of the first.

Lachlan–Woodrow Theorem

Lachlan and Woodrow determined all the countable homogeneous graphs. Apart from trivial cases, these are the *Henson graphs* H_n for $n \geq 3$ and their complements, and the random graph. H_n is the unique countable homogeneous K_n-free graph which embeds all finite K_n-free graphs, where K_n is the complete graph on n vertices.

Take a countable model of set theory. Let X be the set of all sets which do not contain $n-1$ elements mutually comparable by the membership relation; put $x \sim y$ if $x \in y$ or $y \in x$. This graph is isomorphic to H_n. (This is essentially the same as Henson's original construction of his graphs inside R.)

The generic digraph

There is a digraph analogue of R (countable universal homogeneous). Here is an explicit construction of it.

Take our model $\mathbb{N}$ of hereditarily finite set theory. Now put an arc $x \to y$ if $2x \in y$, and $y \to x$ if $2x+1 \in y$.

Given M_0, M_+, M_- with $M_0 \cap (M_+ \cup M_-) = \emptyset$, take

$$x = \sum_{y \in M_-} 2^{2y} + \sum_{y \in M_+} 2^{2y+1} + 2^z$$

where z is sufficiently large. Then there are arcs from elements of M_- to x, and from x to elements of M_+, but none between x and M_0.

If we restrict to the set of natural numbers for which the $(2i)$th and $(2i+1)$st binary digits are not both 1, for all i, we obtain the generic oriented graph.

Set theory with an atom

Take a countable model of set theory with a single atom $\diamondsuit$. Now let M be any set not containing $\diamondsuit$. Putt

$$M_L = \{A \in M : \diamondsuit \notin A\},$$

$$M_R = \{B \setminus \{\diamondsuit\} : \diamondsuit \in B \in M\}.$$

Then neither M_L nor M_R contains $\diamondsuit$.

In the other direction, given two sets P, Q whose elements don't contain $\diamondsuit$, let $(P \mid Q) = P \cup \{B \cup \{\diamondsuit\} : B \in Q\}$. Then $(P \mid Q)$ doesn't contain $\diamondsuit$.

Moreover, for any set M not containing $\diamondsuit$, we have $M = (M_L \mid M_R)$.

Note that any set not containing $\diamondsuit$ can be represented in terms of sets not involving $\diamondsuit$ by means of the operation $(.\mid.)$

For example, $\{\emptyset, \{\diamondsuit\}\}$ is $(\{\emptyset\} \mid \{\emptyset\})$.

The generic digraph again

We define a directed graph as follows:

The vertices are the sets not containing $\diamondsuit$.

If M, N are vertices, then we put an arc $N \to M$ if $N \in M_L$, and an arc $M \to N$ if $N \in M_R$.

Theorem This graph is the generic directed graph.

For if M_+, M^- and M_0 are finite sets of vertices, with $(M_+ \cup M_-) \cap M_0 = \emptyset$, then we can find some z such that $x = (M_- \cup \{z\} \mid M_+)$ has the correct arcs.

The generic poset

The construction of the generic poset is similar to that of the generic digraph just given. We restrict to a sub-collection $\mathcal{P}$ of the sets M not containing $\diamondsuit$ defined by the following recursive properties:

Correctness: $M_L \cup M_R \subseteq \mathcal{P}$ and $M_L \cap M_R = \emptyset$;

Ordering: For all $A \in M_L$ and $B \in M_R$, we have

$$(\{A\} \cup A_R) \cap (\{B\} \cup B_L) \neq \emptyset.$$

Completeness: $A_L \subseteq M_L$ for all $A \in M_L$, and $B_R \subseteq M_R$ for all $B \in M_R$.

Now we put $M \leq N$ if

$$(\{M\} \cup M_R) \cap (\{N\} \cup N_L) \neq \emptyset.$$

Theorem The above-defined structure is isomorphic to the generic poset.

Part of the proof

Note that $\emptyset = (\emptyset \mid \emptyset)$ is in $\mathcal{P}$; the conditions are vacuously satisfied.

First, some notation. We define the *level* $l(M)$ of an element $M \in \mathcal{P}$ by the rules that $l(\emptyset) = 0$ and

$$l(M) = \max\{l(A) : A \in M_L \cup M_R\} + 1$$

for $M \neq \emptyset$.

Also, if $M \neq N$, then any element of $(\{M\} \cup M_R) \cap (\{N\} \cup N_L)$ will be called a *witness* to $M < N$. Note the following:

(a) For $M \in \mathcal{P}$ and $A \in M_L$, $B \in M_R$, we have $A < M < B$.

(b) If W^{MN} is a witness of $M < N$, then $M \leq W^{MN} \leq N$.

(c) If W^{MN} is a witness of $M < N$, then either $l(W^{MN}) < l(M)$ or $l(W^{MN}) < l(N)$.

Part of the proof

Here is the proof of the transitive law. Let $A, B, C \in \mathcal{P}$ satisfy $A < B < C$, and let W^{AB} and W^{BC} be witnesses. First we show that $W^{AB} \leq W^{BC}$. There are four cases:

1. $W^{AB} \in B_L$ and $W^{BC} \in B_R$. Then $W^{AB} < W^{BC}$ by (a).

2. $W^{AB} = B$ and $W^{BC} \in B_R$. Then W^{BC} witnesses $B < W^{BC}$.

3. $W^{AB} \in B_L$ and $W^{BC} = B$. Dual to 2.

4. $W^{AB} = W^{BC} = B$. The result is clear.

In case 4, B witnesses $A < C$. In each of the other cases, if W^{AC} witnesses $W^{AB} < W^{BC}$, then a little argument shows that W^{AC} also witnesses $A < C$.

Surreal numbers

Surreal numbers, as defined by Conway in *On Numbers and Games* (and named by Knuth in *Surreal Numbers*), are objects of the form $X = (X_L \mid X_R)$, where every member of X_L and X_R is a surreal number, and every member of X_L is strictly less than every member of X_R. The ordering is defined by the rule that $A \leq Y$ if and only if $Y \not<_s z$ for all $z \in X_L$ and $z \not<_s X$ for all $z \in Y_R$. (Here we use $<_s$ for the ordering of surreal numbers, to distinguish it from the ordering in the poset $\mathcal{P}$.)

Now it is not too hard to prove the following statements:

- Every element of $\mathcal{P}$ is a surreal number.

- If $M, N \in \mathcal{P}$ and $M < N$, then $M <_s N$. In other words, the ordering on $\mathcal{P}$ induced by Conway's ordering on the surreal numbers is a linear extension of the poset ordering on $\mathcal{P}$.

Finitely presented structures

Hubička and Nešetřil make the following definition:

A countable relational structure S is *finitely presented* if there exists a finitely axiomatisable theory $\mathcal{T}$ and, for each relation R of S, a finitely axiomatisable theory $\mathcal{T}_R$, such that the set of all finite models of $\mathcal{T}$ (in some countable model of set theory) with the relations among them induced by the theories $\mathcal{T}_R$ induce a structure isomorphic to S.

The random graph, Henson's graphs, generic digraph and oriented graphs are all shown to be finitely presented by the descriptions given above. This is not immediate for the generic poset because of the recursive definition of $\mathcal{P}$; but Hubička and Nešetřil give an alternative description showing that it is indeed finitely presented. Indeed, they show that all countable homogeneous graphs and posets are finitely presented.

Homogeneous digraphs

Cherlin determined the countable homogeneous digraphs; there are uncountably many, so they cannot all be finitely presented.

However, most of them are defined by an antichain of forbidden tournaments.

If the antichain is finite, the corresponding homogenous structure is finitely presented (by an argument like that for Henson's graphs).

Furthermore, all countable homogeneous tournaments are finitely presented. (According to a theorem of Lachlan, there are only three of these.)